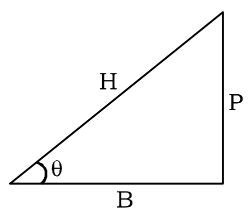
TRIGONOMETRY

TRIGONOMETRIC RATIOS & IDENTITIES

1. The meaning of Trigonometry

Tri	Gon	Metron
↓	↓	↓
3	sides	Measure

Hence, this particular branch in Mathematics was developed in ancient past to measure 3 sides, 3 angles and 6 elements of a triangle. In today's time–trigonometric functions are used in entirely different shapes. The 2 basic functions are sine and cosine of an angle in a right–angled triangle and there are 4 other derived functions.



sin θ	cos θ	tan θ	cot θ	sec θ	cosec θ
$\frac{P}{H}$	$\frac{B}{H}$	$\frac{P}{B}$	$\frac{B}{P}$	$\frac{H}{B}$	$\frac{H}{P}$

2. Basic Trigonometric Identities

(a) $\sin^2\theta + \cos^2\theta = 1 : -1 \le \sin\theta \le 1; -1 \le \cos\theta \le 1 \ \forall\ \theta \in R$

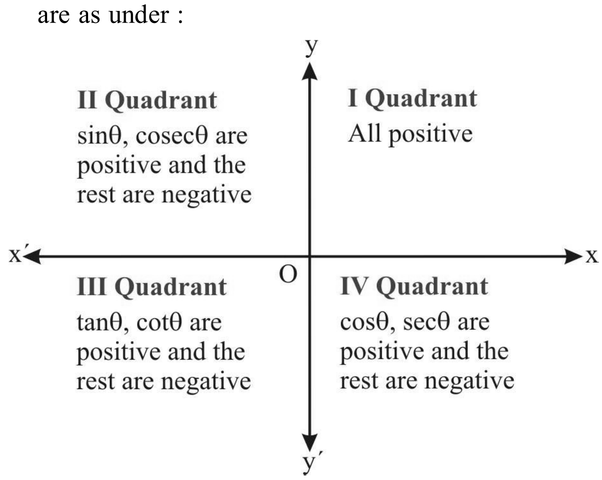
(b) $\sec^2\theta - \tan^2\theta = 1 : |\sec\theta| \ge 1 \ \forall\ \theta \in R$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1 : |\text{cosec}\theta| \ge 1 \ \forall\ \theta \in R$

Trigonometric Ratios of Standard Angles

T–Ratio ↓	0°	30°	45°	60°	90°
			Angle (θ)		
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

The sign of the trigonometric ratios in different quadrants are as under :



3. Trigonometric Ratios of Allied Angles

Using trigonometric ratio of allied angles, we could find the trigonometric ratios of angles of any magnitude.

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$

$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$ $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$

$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$ $\sec(-\theta) = \sec\theta$

$\sec\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta$ $\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)=\sec\theta$

$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$ $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$

$\sin(\pi-\theta)=\sin\theta$ $\cos(\pi-\theta)=-\cos\theta$

$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$ $\cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta$

$\tan(\pi-\theta)=-\tan\theta$ $\cot(\pi-\theta)=-\cot\theta$

$\sec\left(\frac{\pi}{2}+\theta\right)=-\operatorname{cosec}\theta$ $\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\sec\theta$

$\sec(\pi-\theta)=-\sec\theta$ $\operatorname{cosec}(\pi-\theta)=\operatorname{cosec}\theta$

$\sin(\pi+\theta)=-\sin\theta$ $\cos(\pi+\theta)=-\cos\theta$

$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$ $\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$

$\tan(\pi+\theta)=\tan\theta$ $\cot(\pi+\theta)=\cot\theta$

$\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta$ $\cot\left(\frac{3\pi}{2}-\theta\right)=\tan\theta$

$\sec(\pi+\theta)=-\sec\theta$ $\operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta$

$\sec\left(\frac{3\pi}{2}-\theta\right)=-\operatorname{cosec}\theta$ $\operatorname{cosec}\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$

$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$ $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$

$\sin(2\pi-\theta)=-\sin\theta$ $\cos(2\pi-\theta)=\cos\theta$

$\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta$ $\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta$

$\tan(2\pi-\theta)=-\tan\theta$ $\cot(2\pi-\theta)=-\cot\theta$

$\sec\left(\frac{3\pi}{2}+\theta\right)=\operatorname{cosec}\theta$ $\operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$

$\sec(2\pi-\theta)=\sec\theta$ $\operatorname{cosec}(2\pi-\theta)=-\operatorname{cosec}\theta$

$\sin(2\pi+\theta)=\sin\theta$ $\cos(2\pi+\theta)=\cos\theta$

$\tan(2\pi+\theta)=\tan\theta$ $\cot(2\pi+\theta)=\cot\theta$

$\sec(2\pi+\theta)=\sec\theta$ $\operatorname{cosec}(2\pi+\theta)=\operatorname{cosec}\theta$

4. Trigonometric Functions of Sum or Difference of Two Angles

(a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(b) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(c) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(d) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(e) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(f) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(g) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(f) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(h) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \cdot \sin(A-B)$

(i) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)$

(j) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

5. Multiple Angles and Half Angles

(a) $\sin 2A = 2 \sin A \cos A$; $\sin\theta = 2 \sin\dfrac{\theta}{2} \cos\dfrac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta$

(c) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \dfrac{2\tan A}{1-\tan^2 A}$; $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

6. Transformation of Products into Sum or Difference of Sines & Cosines

(a) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

7. Factorisation of the Sum or Difference of Two Sines or Cosines

(a) $\sin C + \sin D = 2 \sin\frac{C+D}{2} \cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2 \cos\frac{C+D}{2} \sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2 \cos\frac{C+D}{2} \cos\frac{C-D}{2}$

(d) $\cos C - \cos D = -2 \sin\frac{C+D}{2} \sin\frac{C-D}{2}$

8. Important Trigonometric Ratios

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in Z$

(b) $\sin 15^\circ$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos\frac{5\pi}{12}$;

$\cos 15^\circ$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin\frac{5\pi}{12}$;

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$;

$\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15^\circ$

(c) $\sin\frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ &

$\cos 36^\circ$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

9. Conditional Identities

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

10. Range of Trigonometric Expression

$E = a \sin\theta + b \cos\theta$

$E = \sqrt{a^2+b^2}\sin(\theta+\alpha), \left(\text{where } \tan\alpha = \frac{b}{a}\right)$

$E = \sqrt{a^2+b^2}\cos(\theta-\beta), \left(\text{where } \tan\beta = \frac{a}{b}\right)$

Hence for any real value of θ, $-\sqrt{a^2+b^2} \le E \le \sqrt{a^2+b^2}$

11. Sine and Cosine Series

(a) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

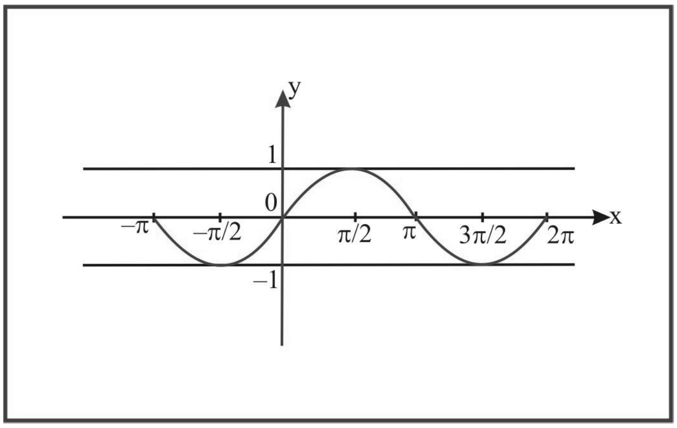
(b) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

12. Graphs of Trigonometric Functions

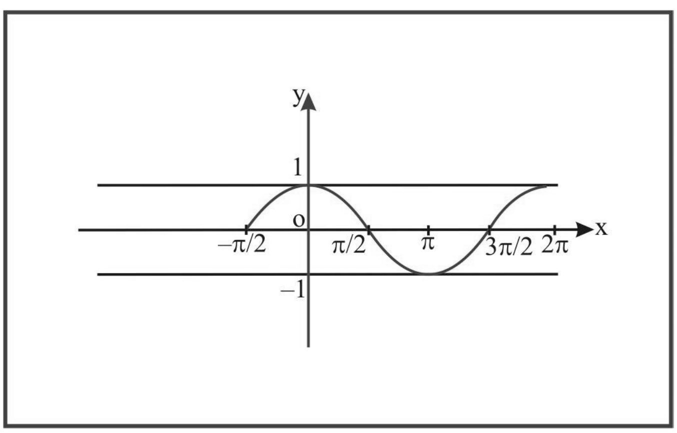
(a) $y = \sin x$,

$x \in R$; $y \in [-1, 1]$



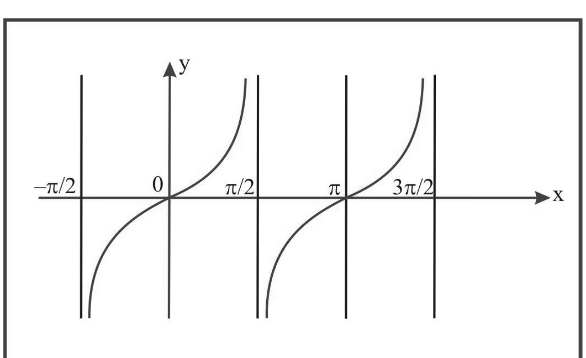
(b) $y = \cos x$,

$x \in R$; $y \in [-1, 1]$



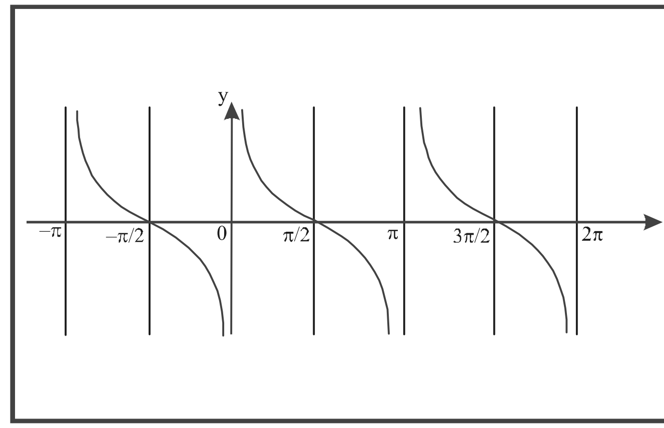
(c) $y = \tan x$,

$$x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in Z\right\};\ y \in R$$



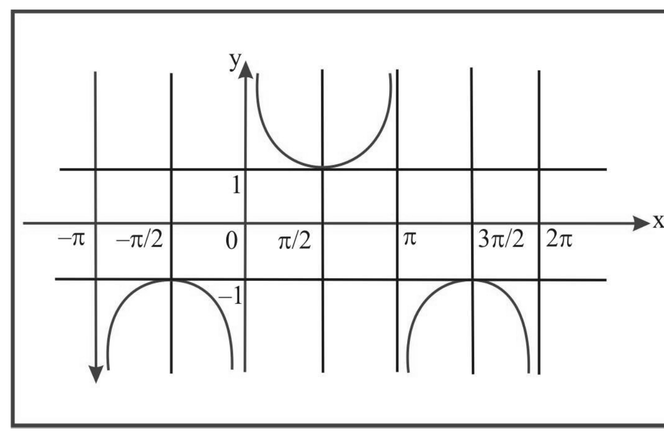
(d) $y = \cot x$,

$x \in R - \{n\pi;\ n \in z\}$; $y \in R$



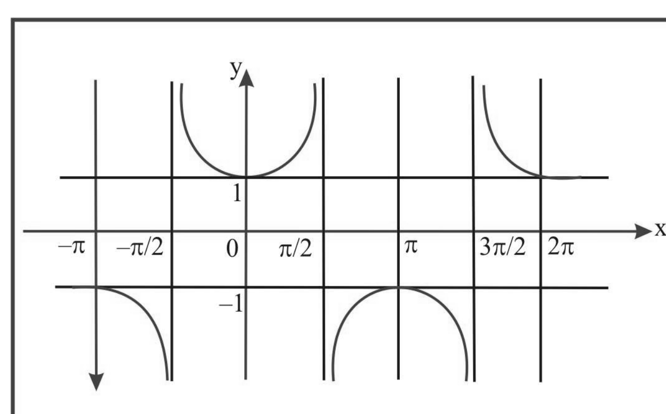
(e) $y = \text{cosec } x$,

$x \in R - \{n\pi;\ n \in Z\}$; $y \in (-\infty, -1] \cup [1, \infty)$



(f) $y = \sec x$,

$$x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in Z\right\};\ y \in (-\infty, -1] \cup [1, \infty)$$



TRIGONOMETRIC EQUATIONS

13. Trigonometric Equations

The equations involving trigonometric functions of unknown angles are known as Trigonometric equations.

e.g., $\cos\theta = 0$, $\cos^2\theta - 4\cos\theta = 1$.

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g., $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$

Thus, the trigonometric equation may have infinite number of solutions and can be classified as :

(i) Principal solution

(ii) General solution

14. General Solution

Since, trigonometric functions are periodic, a solution generalised by means of periodicity of the trigonometrical functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

14.1 Results

1. $\sin\theta = 0 \Leftrightarrow \theta = n\pi$
2. $\cos\theta = 0 \Leftrightarrow \theta\ (2n + 1)\ \frac{\pi}{2}$
3. $\tan\theta = 0 \Leftrightarrow \theta = n\pi$
4. $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
5. $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$.
6. $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
7. $\sin^2\theta = \sin^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$.
8. $\cos^2\theta = \cos^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$.
9. $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$.
10. $\sin\theta = 1 \Leftrightarrow \theta = (4n + 1)\ \frac{\pi}{2}$.
11. $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$.
12. $\cos\theta = -1 \Leftrightarrow \theta = (2n + 1)\pi$.
13. $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha$.

Note...

1. Every where in this chapter 'n' is taken as an integer, if not stated otherwise.
2. The general solution should be given unless the solution is required in a specified interval or range.
3. α is taken as the principal value of the angle. (i.e., Numerically least angle is called the principal value).